

# Improving the fidelity of teleportation through noisy channels using weak measurement

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We employ the technique of weak measurement in order to enable preservation of teleportation fidelity for two-qubit noisy channels. We consider one or both qubits of a maximally entangled state to undergo amplitude damping, and show that the application of weak measurement and a subsequent reverse operation could lead to a fidelity greater than  $2/3$  for any value of the decoherence parameter. The success probability of the protocol decreases with the strength of weak measurement, and is lower when both the qubits are affected by decoherence. Finally, our protocol is shown to work for the Werner state too.

**Keywords :** Teleportation; Entanglement; Fidelity; Weak measurement; Decoherence.

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## I. INTRODUCTION

The primary goal of quantum information processing is to enable performing tasks that are unable to be accomplished classically. Teleportation [1] is a typical information processing task where at present there is intense activity in extending the experimental frontiers [2]. At the practical level teleportation is implemented through the sharing of quantum entanglement by separated parties involving the transmission of quantum particles over large distances. Environmental interaction is a ubiquitous process here, which unless controlled through well-devised means, leads to an inevitable loss of fidelity of the teleported quantum states. Depending upon the magnitude of environmental effects, the fidelity could fall below the maximum limit attainable using classical means, thereby nullifying the quantum advantage of teleportation.

Though decoherence is generally responsible for the decay of quantum correlations in entangled states, and the associated loss of fidelity for the corresponding information processing tasks for which such states are utilized as resources, it has been noted that under certain specially chosen conditions, it could also have a reverse effect. Entanglement between two systems could be created or increased by their collective interactions with a common environment [3]. Applications of such effects in entanglement generation using trapped ions and cavity fields have been suggested [4, 5]. For the specific case of teleportation it has been observed that the effect of amplitude damping on one of the qubits of a shared bipartite state could lead to the enhancement of fidelity above the classical limit for a class of states whose fidelity lies just below quantum region [6]. However, such an improvement is possible only for low values of the damping parameter,

and occurs only for a restricted class of states [7].

The preservation of entanglement in open systems is an important concern, and in the present work we will approach this issue from another suggested direction. Recently, the application of weak measurements has been suggested as a practically implementable method for protecting the fidelity of quantum states undergoing decoherence through the amplitude damping channel [8–13]. The original concept of weak measurements proposed several years ago [14] showed how it would be possible to get an experimental outcome outside the eigenvalue spectrum of an observable, if a sufficiently weak coupling of the system and the apparatus along with the technique of post-selection is employed. The idea of weak measurements has more recently been utilized in several interesting applications such as demonstration of wave particle duality using cavity-QED experiments [15], superluminal propagation of light [16], observations of spin Hall effect [17], trajectories of photons [18], direct measurement of the quantum wave function [19], and measurement of ultrasmall time delays of light [20].

The motivation of this work is to show how the fidelity of teleportation using the resource of two qubits open to amplitude damping environments could be protected with the help of weak measurement. For this purpose we utilize the technique of weak measurement and its reversal as employed recently in order to exhibit the suppression of the effect of amplitude damping decoherence in preserving the entanglement of two-qubit states [8–13]. We first study maximally entangled two-qubit channels which are the most widely used resources in teleportation, and the effects of amplitude damping on which have been investigated earlier for the purpose of obtaining fidelity greater than  $2/3$  without using weak measurement [6, 7]. A fidelity below the classical limit of  $2/3$  can be obtained with the help of shared randomness. It may be noted that with the help of classical resources it is never possible to exceed the limit of  $2/3$  by employing any possible strategy including post-selection [21]. We adapt the technique of weak measurement and its reversal in the

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context of such a setting through the calculation of the optimal strength of weak measurements required in order to maximize the output fidelity. Finally, in order to show that the protocol of improving teleportation fidelity using weak measurement and its reversal is not restricted to the case of pure states, we also present an example of the mixed Werner state.

The plan of this paper is as follows. In Section II we provide a brief review of teleportation through two-qubit amplitude damping channels. In Section III we present our analysis of employing weak measurements in teleportation. As in any protocol involving weak measurement [8–14, 17–20], the role of post-selection is important here, since it allows one to work with only a sub-ensemble of all the initial states. Our analysis based on such post-selection shows that an average fidelity greater than  $2/3$  is attainable for any strength of decoherence for all maximally entangled two qubits if one of them undergoes damping, while such a result holds for a sub-class of maximally entangled two qubits if both of them are affected by decoherence. In Section IV the Werner state is considered when both qubits interact with the environment, and it is shown the teleportation fidelity may be enhanced to the quantum region for a large range of the mixing parameter and the strength of decoherence. We make some concluding remarks in Section V.

## II. TELEPORTATION THROUGH TWO-QUBIT AMPLITUDE DAMPING CHANNELS

In quantum teleportation with the help of entanglement the sender (say, Alice) is able to transfer the unknown quantum state of a qubit to a receiver (say, Bob), stationed at a distant location by performing local quantum operations and communicating two bits of classical information to Bob. The efficiency of teleportation, i.e., closeness of the teleported state with the initial state,  $|\psi_i\rangle$  is determined by the fidelity  $F$  given by [22]

$$F = \langle \psi_i | \sigma(|\psi_f\rangle) | \psi_i \rangle, \quad (1)$$

where  $\sigma(|\psi_f\rangle)$  is the density of the teleported state  $|\psi_f\rangle$ , and the average is taken over all initial states. For a given two-qubit entangled state  $\sigma_{12}$  shared between Alice (who possesses the qubit labeled ‘1’) and Bob (who possesses the qubit labeled ‘2’), the relation of the teleportation fidelity with the fully entangled fraction (FEF),  $f(\sigma_{12})$  of  $\sigma_{12}$  is given by [23]

$$F(\sigma_{12}) = \frac{2f(\sigma_{12}) + 1}{3}, \quad (2)$$

where  $f(\sigma_{12})$  is defined by [24]

$$f(\sigma_{12}) = \max_{|\phi\rangle} \langle \phi | \sigma_{12} | \phi \rangle, \quad (3)$$

with the maximization taken over all two-qubit maximally entangled states  $|\phi\rangle$ . For the shared maximally

entangled states  $\sigma_{12}^M$ ,  $f(\sigma_{12}^M) = 1$  and  $F(\sigma_{12}^M) = 1$ . In absence of entanglement, i.e., by using shared randomness, the average teleportation fidelity achieved is  $2/3$  [25].

Let us suppose that Alice prepares two qubits in one of the four maximally entangled states, given by

$$|\psi_{\pm}^M\rangle = \frac{|00\rangle_{12} \pm |11\rangle_{12}}{\sqrt{2}} \quad (4)$$

$$|\phi_{\pm}^M\rangle = \frac{|01\rangle_{12} \pm |10\rangle_{12}}{\sqrt{2}}, \quad (5)$$

where subscript  $i \in \{1, 2\}$  represents the  $i$ -th qubit, and sends the second qubit to Bob. At the time of transit over the environment, the second qubit interacts with the environment. Due to this interaction, the entanglement between the qubits decreases and the maximally entangled state becomes a mixed state  $\sigma_{12}$ . If the FEF  $f(\sigma_{12}) \leq 1/2$ , the state  $\sigma_{12}$  is useless for the teleportation, as one can achieve the fidelity  $2/3$  on average classically.

In Ref. [6], the authors investigated whether using trace preserving LOCC (local operations and classical communications), one could get the quantum advantage, i.e., the fidelity to lie between  $2/3$  and 1 from the shared entangled state  $\sigma_{12}$  with  $f(\sigma_{12}) \leq 1/2$ . Any bistochastic map ( $\Lambda$ ) which preserves both the trace and identity ( $I$ ), i.e., ( $\Lambda(I) = I$ ) fails to improve the FEF from the classical region ( $0 \leq f \leq 1/2$ ) to the quantum region ( $f > 1/2$ ). Badziag et al. [6] showed that for a class of states  $\rho_{12}$  given by

$$\rho_{12} = \begin{pmatrix} \lambda_{11} & 0 & 0 & \lambda_{14} \\ 0 & \lambda_{22} & -\gamma_{23} & 0 \\ 0 & -\gamma_{23} & \lambda_{33} & 0 \\ \lambda_{14} & 0 & 0 & \lambda_{44} \end{pmatrix}, \quad (6)$$

where  $\gamma_{23} \geq 0$  and  $\lambda_{14}$  are real; and  $\lambda_{22} + \lambda_{33} \geq \frac{1}{2}$ ,  $\gamma_{23} \geq (1 - \lambda_{22} - \lambda_{33})/2$ , the fidelity ( $F(\rho_{12}) = (1 + \lambda_{22} + \lambda_{33} + 2\gamma_{23})/3 \geq 2/3$ ) can be enhanced by applying a non-bistochastic map  $\bar{\Lambda}$ . For the choice of parameters  $\lambda_{11} = \lambda_{14} = 0$ ,  $\lambda_{22} = 3 - 2\sqrt{2}$ ,  $\lambda_{33} = 1$ ,  $\lambda_{44} = 2\sqrt{2} - 2$  and  $\gamma_{23} = \sqrt{2} - 1$ , the fidelity of the above state ( $F(\rho_{12}) = 2/3$ , which belongs to classical region) can be enhanced up to  $\frac{2.06}{3}$  (which lies in the quantum region) by applying  $\bar{\Lambda}$  on any one of the qubits [6]. The map  $\bar{\Lambda}$  which represents the dissipative interaction of any one qubit with the environment via the amplitude damping channel (ADC), is given by

$$\bar{\Lambda}(\rho_{\alpha}) = W_{\alpha,0} \rho_{\alpha} W_{\alpha,0}^{\dagger} + W_{\alpha,1} \rho_{\alpha} W_{\alpha,1}^{\dagger}, \quad (7)$$

where  $\alpha \in \{1, 2\}$ ,  $\rho_{1(2)} = Tr_{2(1)}[\rho_{12}]$ , and the operators  $W_{\alpha,i}$  are given by

$$W_{\alpha,0} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\bar{D}_{\alpha}} \end{pmatrix}, \quad W_{\alpha,1} = \begin{pmatrix} 0 & \sqrt{\bar{D}_{\alpha}} \\ 0 & 0 \end{pmatrix}, \quad (8)$$

where  $\bar{D}_{\alpha} = 1 - D_{\alpha}$ . Here  $D_1$  and  $D_2$  are the strength of interactions of the 1st qubit (belonging to Alice) and

the 2nd qubit (belonging to Bob) with the environment, respectively, and  $\sum_i W_{\alpha,i}^\dagger W_{\alpha,i} = I$ . The above map describes the interaction of the environment (which is initially in the state  $|0\rangle_E$ ) with the qubit by the following transitions

$$\begin{aligned} |0\rangle_i |0\rangle_E &\rightarrow |0\rangle_i |0\rangle_E, \\ |1\rangle_i |0\rangle_E &\rightarrow \sqrt{D_\alpha} |1\rangle_i |0\rangle_E + \sqrt{D_\alpha} |0\rangle_i |1\rangle_E, \end{aligned} \quad (9)$$

where  $i \in \{1, 2\}$  and  $\alpha = 1(2)$  for  $i = 1(2)$ .

Later, in Ref.[7], it was shown that the above interesting class of states  $\rho_{12}$  (used in Ref.[6]) are obtained when Alice prepares the two-qubit maximally entangled state only in the class given by Eq.(4) and sends one qubit (say, the 2nd qubit) to Bob over ADC. Now, for the purpose of enhancing the fidelity, Alice allows her qubit (1st qubit) also to interact with the environment via ADC. According to Bandyopadhyay [7], the fidelity is increased from the classical region to the quantum region due to the enhancement of classical correlations by the application of ADC on the 1st qubit as LOCC by itself is unable to increase the entanglement. This protocol is not effective if the prepared maximally entangled state is chosen from the class given by Eq.(5).

### III. APPLICATION OF WEAK MEASUREMENT AND MEASUREMENT REVERSAL

In earlier studies [8–12] it has been shown that the effect of amplitude damping decoherence (given by Eq.(9)) can be suppressed by weak quantum measurement and reversing quantum measurement (WMRQM) [10]. In the present work we consider two cases. In the 1st case (“*Case I*”), Alice prepares two qubits in one of the above two classes given by Eqs.(4) and (5), and sends the 2nd qubit to Bob. Here the 2nd qubit is affected by ADC and 1st qubit is unaffected. In this case, for all shared states  $\rho_{12}$  whose teleportation fidelities lie in the classical region, Alice and Bob are able to enhance the fidelity to the quantum region with the help of WMRQM, as we show below. In the second case (“*Case II*”), we consider both the 1st and 2nd particles to be interacting with environment. Here we show that for the class of states which are unable to achieve fidelity in the quantum region after allowing the interaction of Alice’s particle with the environment, the help of WMRQM enables attainment of fidelity above classical region. However, if the prepared state is chosen from the class given by Eq.(5), the WMRQM technique fails to shift the fidelity from the classical region to the quantum region. We also calculate the success probability (which is a consequence of the non-unitary operation for the weak measurement) [11] and show how it decreases with increment of the strength of the weak measurement.

Our protocol for both the cases proceeds as follows. First, Alice prepares two qubits in one of the maximally entangled states given by Eqs.(4) and (5). Before allowing the interaction with environment via ADC, Alice makes a weak measurement with the strength  $p_i$  on the  $i$ -th particle ( $i = 1, 2$ ). The weak measurement is achieved by reducing the sensitivity of the detector, i.e., the detector clicks with probability  $p_i$  if the input qubit is in the state  $|1\rangle_i$ , and never clicks if the input qubit is in the state  $|0\rangle_i$  [10, 11]. When the detector clicks, the protocol fails as the input state collapses on the state  $|1\rangle_i$  in an irreversible way. The success probability plays an important role in our protocol. When the detector does not click, the input state partially collapses towards the state  $|0\rangle_i$  which is unaffected by the interaction given by Eq.(9) [10]. The measurement operator corresponding to the detection of the particle is given by

$$M_{\alpha,1} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{p_\alpha} \end{pmatrix}, \quad (10)$$

which does not have any inverse and hence,  $M_{\alpha,1}$  is irreversible. The measurement operator that describes the situation when the detector has not clicked is given by

$$M_{\alpha,0} = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\bar{p}_\alpha} \end{pmatrix}, \quad (11)$$

where  $\bar{p}_\alpha = 1 - p_\alpha$  and  $M_{\alpha,0}^\dagger M_{\alpha,0} + M_{\alpha,1}^\dagger M_{\alpha,1} = I$ . Here,  $M_{\alpha,0}$  is the reversible having a mathematical inverse.

*Case I* : Here, only the 2nd qubit is affected by the amplitude damping decoherence when Alice sends it to Bob over the environment. To reduce the effect of ADC, Alice makes a weak measurement before sending the 2nd qubit and after receiving it, Bob makes a reverse weak measurement. After making the weak measurement on the 2nd qubit by Alice, the two-qubit state (unnormalized) becomes

$$\begin{aligned} \rho_\pm^W &= (I \otimes M_{2,0}) |\psi\rangle_\pm^M \langle\psi| (I \otimes M_{2,0}^\dagger) \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \pm\sqrt{\bar{p}_2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm\sqrt{\bar{p}_2} & 0 & 0 & \bar{p}_2 \end{pmatrix} \end{aligned} \quad (12)$$

and

$$\begin{aligned} \sigma_\pm^W &= (I \otimes M_{2,0}) |\phi\rangle_\pm^M \langle\phi| (I \otimes M_{2,0}^\dagger) \\ &= \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \bar{p}_2 & \pm\sqrt{\bar{p}_2} & 0 \\ 0 & \pm\sqrt{\bar{p}_2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \quad (13)$$

when Alice prepares the initial state in the maximally entangled forms given by Eq.(4) and Eq.(5), respectively. Here the detector’s inefficiency, or the success probability is given by

$$P_2^D = \text{Tr}[\rho_\pm^W] = \text{Tr}[\sigma_\pm^W] = (1 - \frac{p_2}{2}). \quad (14)$$

Next, Alice sends the 2nd qubit to Bob over ADC. Due to the effect of the interaction, the shared state  $\rho_{\pm}^W$  becomes

$$\begin{aligned}\rho_{\pm}^D &= (I \otimes W_{2,0})\rho_{\pm}^W(I \otimes W_{2,0}^\dagger) + (I \otimes W_{2,1})\rho_{\pm}^W(I \otimes W_{2,1}^\dagger) \\ &= \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & k1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & D_2\bar{p}_2 & 0 \\ k1 & 0 & 0 & k1^2 \end{pmatrix},\end{aligned}\quad (15)$$

where  $k1 = \pm\sqrt{D_2\bar{p}_2}$ . Similarly,  $\sigma_{\pm}^W$  becomes

$$\begin{aligned}\sigma_{\pm}^D &= (I \otimes W_{2,0})\sigma_{\pm}^W(I \otimes W_{2,0}^\dagger) + (I \otimes W_{2,1})\sigma_{\pm}^W(I \otimes W_{2,1}^\dagger) \\ &= \frac{1}{2} \begin{pmatrix} D_2\bar{p}_2 & 0 & 0 & 0 \\ 0 & k1^2 & k1 & 0 \\ 0 & k1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}\quad (16)$$

Finally, Bob applies the reverse quantum measurement [10]  $N_{2,0}$  (corresponding to  $M_{2,0}$  given in Eq.(11)) given by

$$N_{2,0} = \begin{pmatrix} \sqrt{\bar{q}_2} & 0 \\ 0 & 1 \end{pmatrix}, \quad (17)$$

where  $q_2$  is the strength of the weak measurement on the 2nd qubit and  $\bar{q}_2 = 1 - q_2$ . At the end, Alice and Bob actually share one of the states given by

$$\rho_{\pm}^R = (I \otimes N_{2,0})\rho_{\pm}^D(I \otimes N_{2,0}^\dagger) \quad (18)$$

$$\begin{aligned}\rho_{\pm}^R &= \begin{pmatrix} \frac{\bar{q}_2}{2} & 0 & 0 & \frac{\pm\sqrt{D_2\bar{p}_2\bar{q}_2}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{D_2\bar{p}_2\bar{q}_2}{2} & 0 \\ \frac{\pm\sqrt{D_2\bar{p}_2\bar{q}_2}}{2} & 0 & 0 & \frac{D_2\bar{p}_2}{2} \end{pmatrix} \\ \sigma_{\pm}^R &= (I \otimes N_{2,0})\sigma_{\pm}^D(I \otimes N_{2,0}^\dagger) \\ &= \begin{pmatrix} \frac{D_2\bar{p}_2\bar{q}_2}{2} & 0 & 0 & 0 \\ 0 & \frac{D_2\bar{p}_2}{2} & \frac{\pm\sqrt{D_2\bar{p}_2\bar{q}_2}}{2} & 0 \\ 0 & \frac{\pm\sqrt{D_2\bar{p}_2\bar{q}_2}}{2} & \frac{\bar{q}_2}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.\end{aligned}\quad (19)$$

The FEFs are equal for both the states given by Eqs.(18) and (19), i.e., the FEF is the same whether Alice prepares the initial two-qubit state in the class given by Eq.(4) or Eq.(5), and it is given by

$$f_1 = \frac{\bar{p}_2 + \bar{q}_2 + 2\sqrt{D_2\bar{p}_2\bar{q}_2} - D_2\bar{p}_2}{2(\bar{p}_2 + \bar{q}_2) - 2D_2q_2\bar{p}_2} \quad (20)$$

The strength of the weak measurement has to be chosen so as to achieve the purpose of the experiment. In Ref.[13] the authors calculate the optimum strength of the weak measurement that maximizes the concurrence of the non-maximally entangled state used by them in order to protect the entanglement from the interaction

of the qubits with environment via ADC. The optimal value of  $q_2$  which maximally protects the fidelity of the unknown teleported state undergoing amplitude damping is obtained by maximizing  $f_1$  (given by Eq.(20)) with respect to  $q_2$ . It turns out that for both the classes of prepared states, the optimal strength,  $q_2^O$  of the reverse measurement is the same, and is given by

$$q_2^O = \frac{3D_2\bar{p}_2 + D_2^2\bar{p}_2^2 + p_2}{(1 + D_2\bar{p}_2)^2}. \quad (21)$$

Note that though the choice of  $q_2 = p_2 + D_2\bar{p}_2$  optimally preserves the entanglement of the maximally entangled state [12, 13], it does not maximize the fidelity of the state passing through the noisy channel. For the choice of initial state from the class given by Eqs.(4) and (5), using Eqs.(20) and (21) one can calculate the optimal FEF which is given by

$$f_1^O = \frac{2 + D_2\bar{p}_2}{2 + 2D_2\bar{p}_2}, \quad (22)$$

where  $f_1^O$  is bounded by 0.75 (occurs for the choice  $D_2 = 1$  and  $p_2 = 0$ ) and 1 (occurs for either  $p_2 = 1$ , or  $D_2 = 0$ ). Here one may note that the optimal teleportation fidelity  $F_1^O (= \frac{2f_1^O+1}{3})$  always belongs to the quantum region ( $> 2/3$ ) irrespective of the strength of decoherence. Due to the weak measurement and the reverse weak measurement, the overall success probability, i.e., the probability of obtaining the state  $\rho_{\pm}^R$  ( $\sigma_{\pm}^R$ ) when Alice prepares the two-qubit state in the class given by Eq.(4) (Eq.(5)) is given by [11]

$$\begin{aligned}P_{Succ}^1 &= Tr[\rho_{\pm}^R] = Tr[\sigma_{\pm}^R] = \frac{1}{2}(\bar{p}_2 + \bar{q}_2^O - D_2\bar{p}_2) \\ &= \frac{(1 - D_2)(1 - p_2)(2 + D_2(1 - p_2))}{2 + 2D_2(1 - p_2)},\end{aligned}\quad (23)$$

where the success probability lies between 0 (which occurs for either  $D_2 = 1$ , or  $p_2 = 1$ , or both) and 1 (which occurs when both  $D_2 = 0$  and  $p_2 = 0$  hold simultaneously).

Now, let us consider the situation when no weak measurement and its reverse is performed. Due to the effect of interaction on the 2nd particle with the environment via ADC, the FEF of the two-qubit state prepared in one of the two classes of maximally entangled states given by Eqs.(4) and (5), is given by[7]

$$\bar{f}_1 = \frac{1}{4} + \frac{1}{2}\sqrt{1 - D_2} + \frac{1}{4}(1 - D_2) \quad (24)$$

and the corresponding fidelity turns out to be  $\bar{F}_1 = (2\bar{f}_1 + 1)/3$ . In the range  $2\sqrt{2} - 2 \leq D_2 \leq 1$ , the teleportation fidelity  $\bar{F}_1$  lies in the classical region, and for others values, i.e.,  $0 \leq D_2 < 2\sqrt{2} - 2$ ,  $\bar{F}_1$  lies in the quantum region. In the figure, FIG. 1 we compare the  $F_1^O$  with the  $\bar{F}_1$ . One sees that for sufficiently strong environmental interaction, the fidelity could fall below the

quantum region without the help of weak measurement. However, as detailed in our protocol above, when one performs weak measurement and its subsequent reversal, the fidelity is preserved above the classical value for all strengths of decoherence. This result holds irrespective of whether the initial state is chosen to belong to the class given by Eq.(4) or by Eq.(5).

It is interesting to note that the role of the reverse weak measurement done by Bob is more important than the weak measurement made by Alice before sending the 2nd particle to Bob over the ADC. To see this point, we consider that Alice sends the 2nd particle to Bob without making any weak measurement on it, i.e.,  $p_2 = 0$ , over the environment. After getting the 2nd particle, Bob makes an optimal weak measurement given by Eq.(17) with  $q_2 = q_2^O$  given by Eq.(21). The optimal FEF in this case is given by

$$f_{12}^O = \frac{2 + D_2}{2 + 2D_2}, \quad (25)$$

which is obtained from Eq.(22) by putting  $\bar{p}_2 = 1$ , and the corresponding success probability is  $\frac{2-D_2-D_2^2}{2(1+D_2)}$ . Here,  $F_{12}^O (= (2f_{12}^O + 1)/3)$  is not only greater than  $\bar{f}_1$ , but, also  $F_{12}^O$  lies in the quantum region, i.e.,  $5/6 \leq F_{12}^O \leq 1$  for all values of the decoherence parameter  $D_2$  which lie in the region given by  $1 \geq D_2 \geq 0$ .

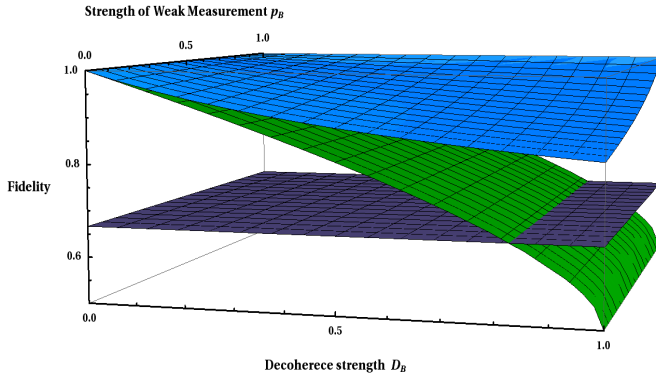


FIG. 1: (Coloronline) The flat plane represents the average classical fidelity  $\frac{2}{3}$ . The surface intersecting it represents the fidelity  $\bar{F}_1$  corresponding to the FEF  $\bar{f}_1$  given by Eq.(24). The uppermost surface represents the fidelity  $F_1^O$  corresponding to the FEF  $f_1^O$  given by Eq.(22).

*Case II.* : In this case, both the 1st and 2nd particles interact with the environment via ADC. To prevent the loss of information about the unknown state in the teleportation protocol, Alice makes weak measurements (given by Eq.(11)), separately on each qubit. When the prepared two-qubit state belongs to the class given by

Eq.(4), after the weak measurement the state becomes

$$\begin{aligned} \rho_{\pm}^{WW} &= (M_{1,0} \otimes M_{2,0})|\psi\rangle_{\pm}^M \langle\psi| (M_{1,0}^{\dagger} \otimes M_{2,0}^{\dagger}) \\ &= \begin{pmatrix} \frac{1}{2} & 0 & 0 & \pm \frac{\sqrt{\bar{p}_1 \bar{p}_2}}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm \frac{\sqrt{\bar{p}_1 \bar{p}_2}}{2} & 0 & 0 & \frac{\bar{p}_1 \bar{p}_2}{2} \end{pmatrix} \end{aligned} \quad (26)$$

Similarly when the state chosen is from the class given by Eq.(5), after weak measurement the state becomes

$$\sigma_{\pm}^{WW} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{\bar{p}_2}{2} & \pm \frac{\sqrt{\bar{p}_1 \bar{p}_2}}{2} & 0 \\ 0 & \pm \frac{\sqrt{\bar{p}_1 \bar{p}_2}}{2} & \frac{\bar{p}_1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (27)$$

The corresponding success probabilities of the weak measurements are given by

$$P_{12}^D(\rho_{\pm}^{WW}) = \text{Tr}[\rho_{\pm}^{WW}] = \frac{1}{2}(1 + \bar{p}_1 \bar{p}_2) \quad (28)$$

and

$$P_{12}^D(\sigma_{\pm}^{WW}) = \text{Tr}[\sigma_{\pm}^{WW}] = \frac{1}{2}(\bar{p}_1 + \bar{p}_2), \quad (29)$$

respectively.

Here, Alice sends the 2nd qubit through the ADC and also allows her qubit (1st qubit) to interact with the environment. Hence, both particles interact with the environment via ADC. After the interaction with the environment, the noisy shared state takes one of the following forms (depending upon the initial state)

$$\rho_{\pm}^{DD} = \begin{pmatrix} \frac{1+D_1 D_2 k_4^2}{2} & 0 & 0 & \pm \frac{k_5 k_4}{2} \\ 0 & \frac{D_1 \bar{D}_2 k_4^2}{2} & 0 & 0 \\ 0 & 0 & \frac{\bar{D}_1 D_2 k_4^2}{2} & 0 \\ \pm \frac{k_5 k_4}{2} & 0 & 0 & \frac{k_5^2 k_4^2}{2} \end{pmatrix} \quad (30)$$

$$\sigma_{\pm}^{DD} = \begin{pmatrix} \frac{D_1 \bar{p}_1}{2} + \frac{D_2 \bar{p}_2}{2} & 0 & 0 & 0 \\ 0 & \frac{\bar{D}_2 \bar{p}_2}{2} & \pm \frac{k_4 k_5}{2} & 0 \\ 0 & \pm \frac{k_4 k_5}{2} & \frac{\bar{D}_1 \bar{p}_1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (31)$$

where  $k_4 = \sqrt{\bar{p}_1 \bar{p}_2}$  and  $k_5 = \sqrt{\bar{D}_1 \bar{D}_2}$ . Next, both Alice and Bob apply reverse weak measurement with the strengths  $q_1$  and  $q_2$  on their respective particles. Let us consider the two classes of initial states separately.

If the initial state is chosen to be from the class given by Eq.(4), the joint state now becomes

$$\rho_{\pm}^{RR} = (N_{1,0} \otimes N_{2,0})\rho_{\pm}^{DD}(N_{1,0}^{\dagger} \otimes N_{2,0}^{\dagger}) \quad (32)$$

where  $N_{2,0}$  is given by Eq.(17) and Alice's reverse weak measurement operator  $N_{1,0}$  is given by

$$N_{1,0} = \begin{pmatrix} \sqrt{q_1} & 0 \\ 0 & 1 \end{pmatrix}. \quad (33)$$

Before maximizing the fidelity  $f(\rho_{\pm}^{RR})$  in this case, for simplicity, let us make the following assumptions. We consider  $D_1 = D_2 = D$ , i.e., both the qubits interact with similar environments, and also,  $p_1 = p_2 = p$ , i.e., the strength of weak measurements on both qubits are the same, and  $q_1 = q_2 = q$ , as well. Similar to ‘Case I’, we maximally enhance the teleportation fidelity (i.e., the FEF  $f(\rho_{\pm}^{RR})$ ) by maximizing  $f(\rho_{\pm}^{RR})$  with respect to the reverse weak measurement strength  $q$ . The optimal FEF is given by

$$f_2^O = f(\rho_{\pm}^{RR}) = \frac{1 + \sqrt{1 + D^2 \bar{p}^2} + D^2 \bar{p}^2}{2(1 + D\bar{p}\sqrt{1 + D^2 \bar{p}^2} + D^2 \bar{p}^2)}, \quad (34)$$

which occurs for the choice

$$q^O = \frac{1 + D^2 \bar{p}^2 - \sqrt{D^2 \bar{p}^2(1 + D^2 \bar{p}^2)}}{1 + D^2 \bar{p}^2}. \quad (35)$$

From the above expression it follows that  $f_2^O$  always lies in the quantum region, i.e., between 0.5 (corresponding to  $D = 1$  and  $p_2 = 0$ ) and 1.0 (corresponding to  $D = 0$  and  $p_2 = 0$ ). Simultaneously, the success probability decreases according to the relation

$$\begin{aligned} P_{Succ}^2 &= Tr[\rho_{\pm}^{RR}] \\ &= \frac{1}{1 + D^2(1-p)^2} ((1-D)^2(1-p)^2(1 + \\ &\quad D(1-p)\sqrt{1 + D^2(1-p)^2} + D^2(1-p)^2)), \end{aligned} \quad (36)$$

where we use  $q = q^O$ . The success probability  $P_{Succ}^2$  varies from 0 to 1.

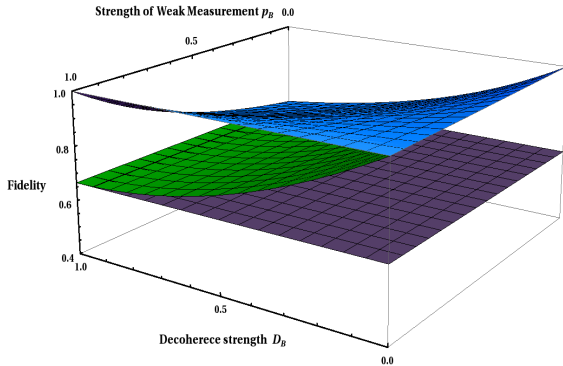


FIG. 2: (Coloronline) The flat plane represents the average classical fidelity  $\frac{2}{3}$ . The lower surface represents the fidelity  $\bar{F}_2$  corresponding to the FEF  $\bar{f}_2$  given by Eq.(37). The upper surface represents the fidelity  $F_2^O$  corresponding to the FEF  $f_2^O$  given by Eq.(34).

Here again, we compare the above situation with the case when decoherence acts without introducing weak measurement and reversal. In the absence of weak measurement, when both the qubits undergo damping, the FEF is given by [7]

$$\bar{f}_2 = 1 - D + \frac{D^2}{2} \quad (37)$$

and  $\bar{F}_2$  is the corresponding fidelity. Note that when  $D$  is chosen in the range  $2\sqrt{2} - 2 \leq D \leq 1$ , though  $\bar{F}_1$  lies in the classical region, it was shown that  $\bar{F}_2$  is quantum [6, 7]. In FIG. 2, we compare the optimal fidelity  $F_2^O$  achieved using weak measurement and reversal with  $\bar{F}_2$ . The comparison shows that the weak measurement technique enhances the fidelity  $F_2^O$  above  $\bar{F}_2$  for the whole range of the decoherence parameter.

Next, we compare the success probabilities for both the cases studied, which are given by Eqs.(23) and (36), respectively. In FIG. 3, we plot the success probabilities  $P_{Succ}^1$  with  $P_{Succ}^2$ , as functions of the decoherence parameter and the strength of weak measurement. Note that in both the cases the corresponding success probabilities fall with the increase of these parameter values. However,  $P_{Succ}^1$  always lies above  $P_{Succ}^2$ , since in the latter case both qubits undergo damping, and two weak measurements are required.

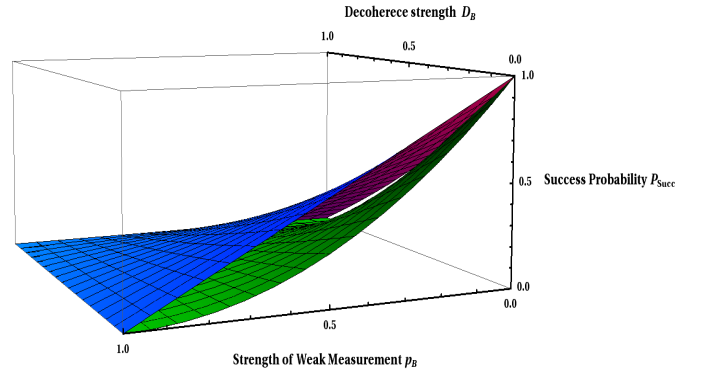


FIG. 3: The upper surface represents the success probability  $P_{Succ}^1$  (given by Eq.(23)) of Case I. The lower surface represents the success probability  $P_{Succ}^2$  (given by Eq.(36)) of Case II.

Let us now consider the situation when Alice prepares the two-qubit state in the class given by Eq.(5). Without applying the weak measurement technique and in the presence of interaction of the environment with both the particles, the FEF  $\bar{f}_2$  is given by [7]  $1 - D \quad \forall D \leq \frac{2}{3}$ , and by  $\frac{D}{2} \quad \forall D \geq \frac{2}{3}$ , where we consider  $D_1 = D_2 = D$ . Comparing with the situation of Case I, when only one of the qubits undergoes damping, one sees that  $\bar{F}_2 \leq \bar{F}_1 \quad \forall \quad 2\sqrt{2} - 2 \leq D \leq \frac{8}{9}$ , and  $\bar{F}_2 > \bar{F}_1 \quad \forall \quad \frac{8}{9} < D \leq 1$ , with  $\bar{F}_2 = \frac{2\bar{f}_2 + 1}{3}$ , and the FEF  $\bar{f}_1$  is given by Eq.(24)).  $\bar{f}_1$  belongs to the classical region, i.e.,  $\bar{f}_1 \leq 1/2$  for the choice of decoherence strength in the range given by  $2\sqrt{2} - 2 \leq D \leq 1$ . When two qubits are allowed to interact with the environment, the fidelity  $\bar{F}_2$  increases, but it never goes to non-classical region for the choice of  $D$  from the region given by  $2\sqrt{2} - 2 \leq D_2 = D \leq 1$ . However, when weak measurement is applied with equal strength on both the qubits for the state prepared in the class given by Eq.(5), it remains unaffected. In order to see this, consider the weak measurement operation described by Eq.(11).

When a weak measurement is performed on a single qubit, say, the 2nd qubit the states given by Eqs. (4) and (5) become  $|\psi_{\pm}^M\rangle^W = M_{2,0}|\psi_{\pm}^M\rangle = \frac{|00\rangle_{12} \pm \sqrt{1-p_2}|11\rangle_{12}}{\sqrt{2-p_2}}$ , and  $|\phi_{\pm}^M\rangle^W = M_{2,0}|\phi_{\pm}^M\rangle = \frac{|01\rangle_{12} \pm \sqrt{1-p_2}|10\rangle_{12}}{\sqrt{2-p_2}}$ , respectively. Note that the states  $|\psi_{\pm}^M\rangle^W$  and  $|\phi_{\pm}^M\rangle^W$  are connected by the unitary rotation  $I \otimes \sigma_x$ . Consequently, our protocol in reducing the effect of decoherence works for both the cases, and the success probability is given by  $\frac{(2-p_2)}{2}$ . Now, if weak measurements are performed on both qubits with equal strength  $p_2$  (i.e.,  $p_1 = p_2$ ), respectively, the states given by Eqs. (4) and (5) in this case become  $|\psi_{\pm}^M\rangle^{WW} = M_{1,0}|\psi_{\pm}^M\rangle^W = \frac{|00\rangle_{12} \pm (1-p_2)|11\rangle_{12}}{\sqrt{1+(1-p_2)^2}}$ , and  $|\phi_{\pm}^M\rangle^{WW} = M_{1,0}|\phi_{\pm}^M\rangle^W = \frac{|01\rangle_{12} \pm |10\rangle_{12}}{\sqrt{2}}$ , respectively. In this case they are not unitarily related, and in fact, the state  $|\phi_{\pm}^M\rangle$  remains unaffected. Hence, the weak measurement technique is not useful for increasing the fidelity beyond the classical region for the state in the class given by Eq.(5).

#### IV. TELEPORTATION USING WERNER STATES

The Werner state is given by

$$\rho_W = \gamma \eta_{\psi_{\pm}} + \frac{1-\gamma}{4}I, \quad (38)$$

where  $\eta_{\psi_{\pm}} = |\psi_{\pm}^M\rangle\langle\psi_{\pm}^M|$ , where  $|\psi_{\pm}^M\rangle$  is given by Eq.(4) and  $\gamma$  is the mixing parameter lying between 0 and 1. In the following analysis we first discuss the effect on the teleportation fidelity due to the decoherence acting on both qubits. We then discuss the case when the technique of weak measurement is applied.

When both qubits interact with the environment via ADC, the FEF of the affected state  $\rho_W^{DD}$  is given by

$$\bar{f}_2(\rho_W^{DD}) = \frac{1}{4} \left( D_2^2(\gamma+1) - 2D_2\gamma - 2(D_2-1)\gamma + \gamma + 1 \right), \quad (39)$$

where  $D_1 = D_2$ , i.e., both qubits interact with the same environment. The corresponding fidelity,  $\bar{F}_2(\rho_W^{DD}) (= \frac{2\bar{f}_2(\rho_W^{DD})+1}{3})$  lies above the classical region for  $0 \leq D_2 \leq \frac{3\gamma-1}{1+\gamma}$  and  $\gamma > \frac{1}{3}$ .

For the prepared state given by Eq.(38), when both qubits are affected by the environment Alice makes a weak measurement on each qubit and then sends the 2nd qubit to Bob. After getting the particle, Alice and Bob both apply reverse weak measurement on their respective particles. At the end they share the state  $\rho_M^{RR}$  which has

FEF given by

$$f(\rho_M^{RR}) = ((q_2-1)^2(D_2^2(p_2-1)^2(\gamma+1) + 2D_2(p_2-1)(\gamma-1) + \gamma+1) - 4(D_2-1)(p_2-1)(q_2-1)\gamma + (D_2-1)^2(p_2-1)^2(\gamma+1)) / (2(q_2^2(D_2^2(p_2-1)^2(\gamma+1) + 2D_2(p_2-1)(\gamma-1) + \gamma+1) + q_2(-2D_2(p_2-1)(p_2\gamma+p_2-2) - 2p_2(\gamma-1) - 4) + p_2^2(\gamma+1) - 4p_2+4)), \quad (40)$$

where  $D_1 = D_2$ ,  $p_1 = p_2$  and  $q_1 = q_2$ . To get the optimal fidelity  $F^O(\rho_M^{RR})$ , we maximize  $f(\rho_M^{RR})$  over the strength of the reverse weak measurement  $q_2$ . The optimal strength is given by

$$q_2^O = \left( D_2^2(p_2-1)^2(\gamma+1) - ((D_2-1)^2(p_2-1)^2(\gamma+1) + (D_2^2(p_2-1)^2(\gamma+1) + 2D_2(p_2-1)(\gamma-1) + \gamma+1))^{1/2} + 2D_2(p_2-1)(\gamma-1) + \gamma+1 \right) / \left( D_2^2(p_2-1)^2(\gamma+1) + 2D_2(p_2-1)(\gamma-1) + \gamma+1 \right). \quad (41)$$

The fidelity  $F^O(\rho_M^{RR})$  lies above the classical region if  $\gamma > \frac{1+D_2-D_2p_2}{3-D_2+D_2p_2}$  ( $D_2 < \frac{1-3\gamma}{(1+\gamma)(p_2-1)}$ ). The corresponding success probability is given by

$$P_{Succ} = \frac{1}{4} ((q_2^O)^2(D_2^2(p_2-1)^2(\gamma+1) + 2D_2(p_2-1)(\gamma-1) + \gamma+1) + q_2^O(-2D_2(p_2-1)(p_2\gamma+p_2-2) + p_2(2-2\gamma)-4) + p_2^2(\gamma+1) - 4p_2+4). \quad (42)$$

When  $D_2 < \frac{1-3\gamma}{(1+\gamma)(p_2-1)}$ , the fidelity  $F^O(\rho_M^{RR}) > \bar{F}_2(\eta_M^{DD})$ , the weak measurement protocol is able to enhance the teleportation fidelity. In Fig.(4) we plot the fidelity  $F^O(\rho_M^{RR})$  versus the Werner state parameter  $\gamma$  and the decoherence strength  $D_2$  for a particular value  $p_2 = 0.4$  of the strength of the weak measurement. Note that even for large values of the mixing parameter  $\gamma$  and the magnitude of decoherence  $D_2$ , the fidelity in the quantum region is achieved through the technique of weak measurement.

#### V. CONCLUSIONS

To summarize, in the present work, we propose a method for maintaining teleportation fidelity over the classical region through noisy channels using the technique of weak measurements. We reduce the loss of information about the unknown state due to interaction with the environment via amplitude damping channel with the help of weak measurement and reversal of weak measurement. We find the optimal strength of reversing the measurement for which the loss is minimum. Our results are exemplified by two classes of two-qubit states, *viz.* maximally entangled pure states, as well as the mixed



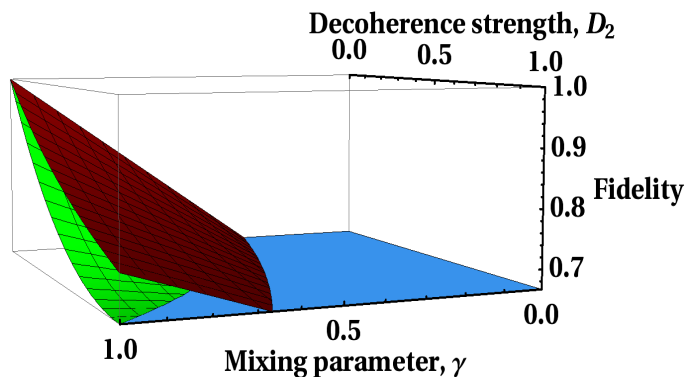


FIG. 4: The upper surface represents the teleportation fidelity when the technique of weak measurement is applied. The lower surface represents the fidelity when weak measurement is not applied.

Werner state. For the prepared two-qubit states given by Eqs.(4) and (5), we show that when only one particle (say, the 2nd qubit which Alice sends to Bob over the environment) interacts with environment, the weak measurement technique is able to enhance the teleportation fidelity arbitrarily close to 1. This result holds good for all maximally entangled states, as well as for all values of the decoherence parameter. In this case even without performing the weak measurement before sending the 2nd particle, i.e.,  $p_2 = 0$ , Bob is able to enhance the fidelity to the quantum region by making a weak

measurement with strength given by Eq.(21). However, without applying the weak measurement technique by Alice and Bob, the teleportation fidelity lies in the classical region for the choice of decoherence strength chosen from the region  $2\sqrt{2} - 2 \leq D_2 \leq 1$ .

Next, when the environment effects both the particles, the weak measurement technique protects the information for the initially prepared state given by Eq.(4), but fails to do so for the state given by Eq.(5). Note that though the states Eq.(4) and Eq.(5) are unitarily equivalent to each other, the nature of post-selection employed by us in the process of the weak measurement is unable to impact the state in the latter case due to its chosen structure. We also show that by increasing the strength of weak measurement, the success probability (which arises as a consequence of the cancellation of the protocol when the detector clicks) decreases. The success of enhancing teleportation fidelity is larger when one qubit interacts compared to the case when both qubits interact with the environment. We also employ our protocol for the Werner state and show that for a large range of the mixing parameter as well as the decoherence strength, the technique of weak measurement is able to improve the teleportation fidelity beyond the classical region.

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